z-Transform

z-Transform

The z-transform is the most general concept for the transformation of discrete-time series.

The Laplace transform is the more general concept for the transformation of continuous time processes.

For example, the Laplace transform allows you to transform a differential equation, and its corresponding initial and boundary value problems, into a space in which the equation can be solved by ordinary algebra.

The switching of spaces to transform calculus problems into algebraic operations on transforms is called operational calculus. The Laplace and z transforms are the most important methods for this purpose.

The Transforms

The Laplace transform of a function *f*(*t*):

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

The one-sided z-transform of a function *x*(*n*):

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

The two-sided z-transform of a function *x*(*n*):

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Relationship to Fourier Transform

Note that expressing the complex variable *z* in polar form reveals the relationship to the Fourier transform:

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)(re^{i\omega})^{-n}, \text{ or}$$

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-i\omega n}, \text{ and if } r = 1,$$

$$X(e^{i\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$$

which is the Fourier transform of $x(n)$.

Region of Convergence

The z-transform of *x*(*n*) can be viewed as the Fourier transform of *x*(*n*) multiplied by an exponential sequence r^{-n}, and the z-transform may converge even when the Fourier transform does not.

By redefining convergence, it is possible that the Fourier transform may converge when the z-transform does not.

For the Fourier transform to converge, the sequence must have finite energy, or:

$$\sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right| < \infty$$

Convergence, continued

The power series for the z-transform is called a Laurent series:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The Laurent series, and therefore the z-transform, represents an analytic function at every point inside the region of convergence, and therefore the z-transform and all its derivatives must be continuous functions of *z* inside the region of convergence.

In general, the Laurent series will converge in an annular region of the z-plane.

Some Special Functions

First we introduce the Dirac delta function (or unit sample function):

$$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \quad \textbf{or} \quad \delta(t) = \begin{cases} 0, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

This allows an arbitrary sequence *x*(*n*) or continuous-time function *f*(*t*) to be expressed as:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$$f(t) = \int_{-\infty}^{\infty} f(x)\delta(x-t)dt$$

Convolution, Unit Step

These are referred to as discrete-time or continuous-time convolution, and are denoted by:

$$x(n) = x(n) * \delta(n)$$

$$f(t) = f(t) * \delta(t)$$

We also introduce the unit step function:

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{or} \quad u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Note also:

$$u(n) = \sum_{k=-\infty}^{\infty} \delta(k)$$

Poles and Zeros

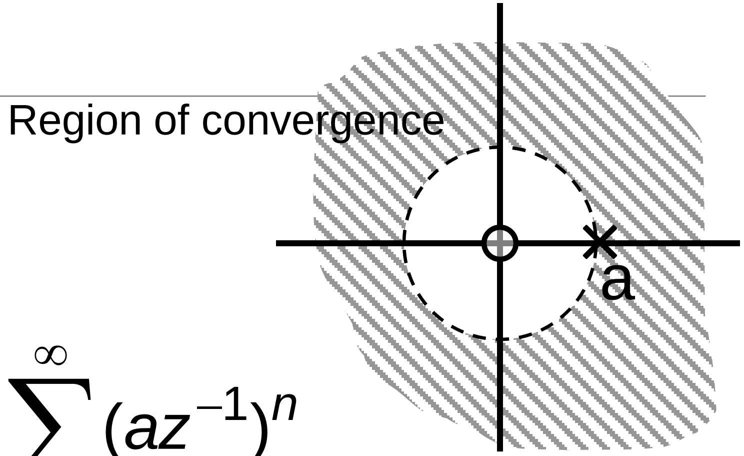
When $X(z)$ is a rational function, i.e., a ration of polynomials in z, then:

1. **The roots of the numerator polynomial are referred to as the zeros of $X(z)$, and**
2. **The roots of the denominator polynomial are referred to as the poles of $X(z)$.**

Note that no poles of $X(z)$ can occur within the region of convergence since the z-transform does not converge at a pole.

Furthermore, the region of convergence is bounded by poles.

Example



$$x(n) = a^n u(n)$$

The z-transform is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

Which converges to:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \quad for \quad |z| > |a|$$

Clearly, *X*(*z*) has a zero at *z* = 0 and a pole at *z* = *a*.

Convergence of Finite Sequences

Suppose that only a finite number of sequence values are nonzero, so that:

$$X(z) = \sum_{n=n_1}^{n_2} x(n) z^{-n}$$

Where n_1 and n_2 are finite integers. Convergence requires

$$|x(n)| < \infty \quad for \quad n_1 \le n \le n_2.$$

So that finite-length sequences have a region of convergence that is at least $0 < |z| < \infty$, and may include either $z = 0$ or $z = \infty$.

Inverse z-Transform

The inverse z-transform can be derived by using Cauchy's integral theorem. Start with the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Multiply both sides by z^{k-1} and integrate with a contour integral for which the contour of integration encloses the origin and lies entirely within the region of convergence of $X(z)$:

$$\frac{1}{2\pi i}\oint_C X(z) z^{k-1} dz = \frac{1}{2\pi i}\oint_C \sum_{n=-\infty}^{\infty} x(n) z^{-n+k-1} dz$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi i}\oint_C z^{-n+k-1} dz$$

$$\frac{1}{2\pi i}\oint_C X(z) z^{k-1} dz = x(n) \text{ is the inverse z-transform.}$$